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Homework #6: Proofs

1. Write a formal proof for each of the following statements:

(a) The product of an odd integer and an even integer is always even.

Proof:

By definition of even, $n = 2k$ for some $k \in \mathbb{Z}$.

By definition of odd, $m = 2L + 1$ for some $L \in \mathbb{Z}$.

Therefore the product of $m \cdot n = 2k \cdot (2L + 1)$

Factoring out 2 results in $2(k \cdot L + 1)$

$k \cdot L + 1$ is an integer, therefore by definition any integer multiplied by 2 is even

Thus, $m \cdot n$ is even.

(b) If one integer is even and another is divisible by 3, then their product is divisible by 6.

Let $n \in \mathbb{Z}$, and n is divisible by 3 $\Rightarrow$ $n = 3L$

By definition of even, $m = 2k$ for some $k \in \mathbb{Z}$

Therefore the product of $m \cdot n = 2k \cdot 3L$

By the distributive property, $m \cdot n = 6(k \cdot L)$

By closure under addition, $m \cdot n = 6(s)$

Thus, by the definition of divisibility, $m \cdot n$ is divisible by 6.

2. For each of the following statements, write the contrapositive statement, and then prove the original by proving its contrapositive.

(a) If $3n$ is odd, then n is odd.

$\rightarrow$ If n is even, then $3n$ is even.

Assume that n is even.

By definition of even, $n = 2k$ for some $k \in \mathbb{Z}$.

Therefore the product of $n \cdot 3 = 3 \cdot 2k$

Factoring out 2 results in $3n = 2(3k)$

$3k$ is an ~~integer~~ integer, therefore by definition any integer multiplied by 2 is even.

Thus, $3n$ is even when n is even.

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2 cont.

(b) If $m + n$ is odd, then m or n must be even.

→ If m and n must be odd, then $m + n$ is even.

Assume m and n are odd

By definition of odd, $m = 2K + 1$ for some $K \in \mathbb{Z}$.

By definition of odd, $n = 2L + 1$ for some $L \in \mathbb{Z}$.

Therefore the product of $m + n$ is $(2K + 1) + (2L + 1)$.

Which can be simplified to $2K + 2L + 2$.

Factoring out 2 results in $m + n = 2(K + L + 1)$

By definition of even, $m + n$ is even.

3. Prove that if n is not divisible by 3, then $n^2 + 2$ is divisible by 3.

Case 1: n is odd

Let an odd integer n not divisible by 3 be given

By definition of odd, $n = 3K + 1$ for some $K \in \mathbb{Z}$.

By substitution

$(n^2 + 2) = (3K + 1)^2 + 2 = (9K^2 + 6K + 1) + 2 = 9K^2 + 6K + 3$

Factoring out 3 results in $3(3K^2 + 2K + 1)$

By closure under multiplication and addition, $n^2 + n = 3m$ where $m \in \mathbb{Z}$

By definition of divisibility, $n^2 + 2 = 3m$ is divisible by 3.

Case 2: n is even

Let an even integer n not divisible by 3 be given

By definition of even, $n = 3K + 2$ for some $K \in \mathbb{Z}$.

By substitution

$(n^2 + 2) = (3K + 2)^2 + 2 = (9K^2 + 12K + 4) + 2 = 9K^2 + 12K + 6$

Factoring out 3 results in $3(3K^2 + 2K + 1)$

By closure under multiplication and addition, $n^2 + n = 3m$ where $m \in \mathbb{Z}$.

By definition of divisibility, $n^2 + 2 = 3m$ is divisible by 3

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Homework # 6: Proofs

4. Give element-wise proofs for each of the following:

(a) If $A \cap B = A$ and $B \cap C = B$, then $A \cap C = A$

Let $a \in A \cap C$

By definition of intersection $a \in A$

Let $b \in A$, since $A \cap B = A$

By assumption $b \in A \cap B$ therefore $b \in B$

Since $B \cap C = B$, by assumption $b \in B \cap C$

Therefore $b \in C$

Thus both $b \in A$ and $b \in C$, so $b \in A \cap C$

(b) If $A \cup B = B$ and $B \cup C = C$, then $A \cup C = C$

Let $a \in A \cup C$

If $a \in A$

then $a \in A \cup B$, since $A \cup B = B$ then $a \in B$

If $a \in B$ then $a \in B \cup C$, since $B \cup C = C$ then $a \in C$

Since anything in $A \cup C$ is in A or C this proves everything in $A \cup C$ is also in C.

If $b \in C$, then $b \in A \cup C$ by definition of union

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Homework #6: Proofs

6. Prove or give a counterexample:

(a) If R is a reflexive relation on A, then $R \circ R$ is a reflexive relation on A.

Let $a \in A$ be given.

Since R is reflexive $(a,a) \in R$.

Due to composition, $(a,a) \in R$ and $(a,a) \in R$ means that (a,a) in $R \circ R$.

Since $(a,a) \in R \circ R$, $\forall a \in A$, relation $R \circ R$ is reflexive.

(b) If R is an antisymmetric relation on A, then $R \circ R$ is an antisymmetric relation on A.

Let $A = \{1,2,3,4\}$

Let $R = \{(1,1),(2,3),(3,4),(4,1)\}$

Due to composition $(1,3) \in R \circ R$, $(2,4) \in R \circ R$, $(3,1) \in R \circ R$, $(4,2) \in R \circ R$.

Since $(1,3) \in R \circ R$ and $(3,1) \in R \circ R$, $1 \neq 3$ $R \circ R$ is not antisymmetric.

(c) If R is a transitive relation on A, then $R \circ R$ is a transitive relation on A.

Let $(a,b) \in R \circ R$ and $(b,c) \in R \circ R$

Due to composition $(a,b_1) \in R$ & $(b_1,b) \in R$